

Model order reduction of commensurate linear discrete-time fractional-order systems

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Abstract: This paper presents an approach to model order reduction of linear time-invariant discrete-time commensurate fractional-order state space systems by means of the Balanced Truncation Approximation and Singular Perturbation Approximation methods. Mathematical formulas for computation of controllability and observability Gramians for the fractional-order systems are derived. This enables an extension of classical SVD-originated model order reduction algorithms to model reduction of discrete-time fractional-order state space systems. Simulation experiment illustrates the efficiency of the introduced methodology.

Keywords: fractional-order system, commensurate-order, controllability and observability Gramians, model order reduction, balanced truncation approximation

1. INTRODUCTION

Modeling, analysis and control of fractional-order dynamics have attracted increasing attention from theoretical and practical points of view. This is due to the properties of various complex processes which can be more suitably modeled by fractional-order differential/integral equations (Tarasov, 2011).

The main problem occurred during implementation of continuous-time fractional-order systems is an infinite sum required to calculate of fractional-order derivatives. Similarly, fractional-order difference requires calculating the sum from 0 to $t+1$, so each incoming sample increases complexity, leading to computational explosion for $t \rightarrow \infty$. For these reasons, in practical implementations of fractional-order systems there are used integer-order approximators of fractional-order derivatives (or differences) obtained in various ways (Ferd, 2006; Dhabale et al., 2015; Kumar and Rawat, 2015; Rydel et al., 2017). The second possibility is an approximation of the whole fractional-order systems (Podlubny, 1999; Monje et al., 2010; Stanisławski and Latawiec, 2016; Baranowski et al., 2016; Stanisławski et al., 2017). In both approaches, we usually obtain very high integer-order model and then classical model reduction techniques are used to reduce the system's order (Krajewski and Viaro, 2014; Stanisławski et al., 2017). Therefore, model order reduction (MOR) of fractional-order systems is usually associated with order reduction for fractional-order approximators.

This paper deals with the problem of model reduction for discrete-time commensurate fractional-order systems but

as a result we obtain lower dimensions of fractional-order model while the fractional order itself remains the same as for the original system. This problem has not been systematically studied until now and only a few papers have appeared for continuous-time fractional-order systems (Tavakoli-Kakhki and Haeri, 2009; Jiang and Xiao, 2015). In this paper, we extend the SVD-based reduction methods that have been previously introduced for classical integer-order systems to discrete-time fractional-order systems.

The paper is organized as follows. Having introduced the problem of fractional-order system approximation in Section 1, a representation of discrete-time commensurate fractional-order state space systems is presented in Section 2. This section also contains fundamentals of model order reduction, particularly for the Balanced Truncation Approximation (BTA) and Singular Perturbation Approximation (SPA) methods. Section 3 presents the main result of the paper in terms of computation of fractional-order controllability and observability Gramians. Numerical examples of Section 4 confirm the ability for use of the introduced Gramians in the reduction process for fractional-order systems. Conclusions of Section 5 complete the paper.

2. PRELIMINARIES

2.1 System Representation

Consider a stable commensurate fractional-order discrete-time LTI MIMO state space system

$$\begin{aligned}\Delta^\alpha x(t+1) &= A_f x(t) + Bu(t), & x_0 \\ y(t) &= Cx(t) + Du(t)\end{aligned}\quad (1)$$

* This work was supported by the Polish National Science Centre on the base of decision no. DEC-2017/01/X/ST7/00885

where $t = 0, 1, \dots$, $x(t) \in \mathfrak{R}^n$ is the state vector, $u(t)$ and $y(t)$ are the input and output signals, respectively, $A_f \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times n_u}$, $C \in \mathfrak{R}^{n_y \times n}$, $D \in \mathfrak{R}^{n_y \times n_u}$ describe the system properties and $\alpha \in (0, 2)$.

Fractional-order difference $\Delta^\alpha x(t+1)$ can be represented by the well known Grünwald-Letnikov fractional-order difference (FD) (Monje et al., 2010)

$$\Delta^\alpha x(t+1) = \sum_{j=0}^{t+1} (-1)^j \binom{\alpha}{j} x(t-j+1) \quad t = 0, 1, \dots \quad (2)$$

with

$$\binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & j > 0 \end{cases}$$

An attempt at implementation of the fractional-order difference implies that the sum is calculated from 0 to $t+1$, which results in increasing computational burden of Eqn. (2) at each time step. Finally, for $t \rightarrow \infty$ Eqn. (2) leads to computational explosion. Therefore, finite-length expansions must be used in practical implementations. For instance, the truncated or finite fractional difference (FFD) can be used as an approximator to FD (see Refs. Monje et al. (2010); Podlubny (1999); Stanisławski and Latawiec (2012)).

$$\Delta^\alpha x(t+1) \cong \sum_{j=0}^L (-1)^j \binom{\alpha}{j} x(t-j+1) \quad t = 0, 1, \dots \quad (3)$$

with $x(l) = 0 \forall l < 0$ and L being an implementation length. It is important to note that accurate approximation of the FD with the FFD requires a very high implementation length L , especially for lower values of fractional orders α . A detailed analysis of approximation accuracy for FFD is presented in Stanisławski et al. (2015).

2.2 Model Order Reduction

An accurate description of complex systems can lead to a high complexity of generated mathematical models. This is particularly evident for models obtained by the use of the Finite Element Method, linearized models of nonlinear systems and integer-order approximations of fractional-order systems. Simulations and controller designs for those large-scale systems are difficult due to high demands for computing power. For this reason, an ability for proper reduction of model complexity, without the loss of dominant dynamic properties, is highly significant.

Consider a discrete-time high-order (integer-order) state space dynamical system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (4)$$

where $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times n_u}$, $C \in \mathfrak{R}^{n_y \times n}$, $D \in \mathfrak{R}^{n_y \times n_u}$ denote the system matrices.

Order reduction for that model can be performed in various ways by the use of the whole series of reduction techniques (Antoulas, 2005; Benner et al., 2017; Rydel and Stanisławski, 2018). Among the reduction methods, very popular are SVD-based methods, which were introduced in the Moore's works (Moore, 1981). An important consideration in these methods is the ability to classify a degree

of reachability and observability of states in the system. This is connected with controllability and observability Gramians of the system.

A definition of the controllability Gramian P is related with the minimal energy required for the transfer of the system from state $x(0) = 0$ to $x(t) = x_p$.

$$\|u_p\|^2 = x_p^T P^{-1} x_p$$

For discrete-time systems with the initial condition $x(0) = 0$, the input in the form of Kronecker delta $\delta(t)$ results in the state

$$\xi(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ A^{t-1} B & \text{for } t > 0 \end{cases}$$

and the finite controllability Gramian at the time $t < \infty$ of this system is defined as

$$P(t) = \sum_{i=0}^t \xi(i) \xi^T(i) \quad (5)$$

The definition of observability Gramian is connected with energy produced by the state $x(0) = x_0$

$$\|y_0\|^2 = x_0^T Q x_0$$

In the absence of the input signal $u(t) = 0$, the output of the model, given by the nonzero initial conditions, is as follows

$$y(t) = \eta(t)x(0)$$

where

$$\eta(t) = \begin{cases} 0 & \text{for } t < 0 \\ CA^t & \text{for } t \geq 0 \end{cases}$$

The finite observability Gramian at the time $t < \infty$ is calculated in the following way

$$Q(t) = \sum_{i=0}^t \eta^T(i) \eta(i) \quad (6)$$

The controllability and observability Gramians for stable systems are positive definite, thus it can be shown that

$$P(t_2) \geq P(t_1) \quad Q(t_2) \geq Q(t_1) \quad \text{for } t_2 \geq t_1 \quad (7)$$

Therefore, the minimal energy for the transfer of the system from the state $x(0) = 0$ to x_p and the largest energy produced by the the state $x(0) = x_0$ is obtained for $t \rightarrow \infty$. For stable discrete-time integer-order systems, the controllability and observability Gramians are finally defined as

$$\begin{aligned} P &= \sum_{i=0}^{\infty} \xi(i) \xi^T(i) = \sum_{i=0}^{\infty} A^i B B^T (A^T)^i \\ Q &= \sum_{i=0}^{\infty} \eta^T(k) \eta(k) = \sum_{i=0}^{\infty} (A^T)^i C^T C A^i \end{aligned} \quad (8)$$

The impact of the particular state variable on the input-output properties of the system is only reflected for a model in the balanced realization form. The state space realization of the system is called balanced, when the controllability and observability Gramians of that system are the same and equal to a diagonal matrix with decreasing Hankel Singular Values (HSV) on the main diagonal. In the purpose to obtain the balanced system, the linear state transformation $x \rightarrow Tx$ is necessary to apply

$$TPT^T = (T^T)^{-1}QT^{-1} = \text{diag}(\sigma_1, \dots, \sigma_n) \quad (9)$$

where σ_i , $i = 1, \dots, n$, are the square roots of the eigenvalues for the product of the controllability and the observability Gramians.

Table 1. Square root algorithm

1. Compute a Cholesky factorization of controllability and observability Gramians $P = S^T S \quad Q = R^T R$
2. Compute the singular value decomposition $RS^T = U\Sigma V$ where: $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$
3. Compute transformation matrices $T = \Sigma^{-\frac{1}{2}} U^T R \quad T^{-1} = S^T V \Sigma^{-\frac{1}{2}}$
4. Compute state space matrices of balanced realization $\bar{A} = TAT^{-1} \quad \bar{B} = TB \quad \bar{C} = CT^{-1} \quad \bar{D} = D$

Determination of the transformation matrix T is not a unique operation and there exist various algorithms described in the literature (Moore, 1981; Laub et al., 1987; Safonov and Chiang, 1989; Varga and Anderson, 2003). The exemplary algorithm is presented in Table 1. The states that are difficult to reach (i.e. require a large amount of energy to reach) and difficult to observe (i.e. yield small amounts of observation energy) correspond to low eigenvalues of controllability and observability Gramians. On this basis, reduction eliminates states that have a weak impact on the system in the sense of controllability and observability Gramians.

Consider the state vector of the balanced realization in the form $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$, where $\bar{x}_1 \in \mathfrak{R}^k$, $\bar{x}_2 \in \mathfrak{R}^{(n-k)}$ are the states corresponding to the Hankel singular values $\sigma_1, \dots, \sigma_k$ and $\sigma_{k+1}, \dots, \sigma_n$, so that $\sigma_1 \geq \dots \geq \sigma_k > \sigma_{k+1} \geq \dots \geq \sigma_n > 0$. The partition of balanced realization into two subsystems can be presented as follows

$$\begin{bmatrix} \bar{x}_1(t+1) \\ \bar{x}_2(t+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u(t)$$

$$y = [\bar{C}_1 \quad \bar{C}_2] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \bar{D} u(t) \quad (10)$$

Various ways of removing the states \bar{x}_2 , which have a negligible influence on input-output properties of the model, result in various model reduction methods. The Balanced Truncation Approximation (BTA) method determines the dominant part of the model by cutting off the state-space matrices of a model, which simply implies that $\bar{x}_2 = 0$. Consequently, the reduced order system of order $k < n$ is given in the following way

$$\begin{aligned} x_r(t+1) &= \bar{A}_{11} x_r(t) + \bar{B}_1 u(t) \\ y_r(t) &= \bar{C}_1 x_r(t) + \bar{D} u(t) \end{aligned} \quad (11)$$

This method does not guarantee preservation of the steady state gain, whereas the modeling accuracy at high frequencies is satisfactory.

The Singular Perturbation Approximation (SPA) method considers the first-order difference $\Delta \bar{x}_2 = \bar{x}_2(t+1) - \bar{x}_2(t) = 0$, which implies that state variables \bar{x}_2 are equal to

$$\bar{x}_2(t) = (I - \bar{A}_{22})^{-1} \bar{A}_{21} \bar{x}_1(t) + (I - \bar{A}_{22})^{-1} \bar{B}_2 u(t)$$

Therefore, the reduced system is given by

$$\begin{aligned} x_r(t+1) &= A_r x_r(t) + B_r u(t) \\ y_r(t) &= C_r x_r(t) + D_r u(t) \end{aligned} \quad (12)$$

where

$$\begin{aligned} A_r &= \bar{A}_{11} + \bar{A}_{12} (I - \bar{A}_{22})^{-1} \bar{A}_{21} \\ B_r &= \bar{B}_1 + \bar{A}_{12} (I - \bar{A}_{22})^{-1} \bar{B}_2 \\ C_r &= \bar{C}_1 + \bar{C}_2 (I - \bar{A}_{22})^{-1} \bar{A}_{21} \\ D_r &= \bar{D} + \bar{C}_2 (I - \bar{A}_{22})^{-1} \bar{B}_2 \end{aligned}$$

In this case, the steady state gain of the reduced model is preserved.

3. MAIN RESULTS

In this Section, the controllability and observability Gramians of the integer-order system are extended to the discrete-time commensurate fractional-order systems.

Theorem 1. Consider a stable discrete-time commensurate fractional-order state space system as in Eqn. (1) with fractional-order difference defined in Eqn. (2). Then, the controllability and observability Gramians of the system can be calculated in the following way

$$P = \sum_{t=0}^{\infty} \phi(t) B B^T \phi(t)^T$$

$$Q = \sum_{t=0}^{\infty} \phi(t)^T C^T C \phi(t)$$
(13)

where $\phi(t)$, $t = 0, 1, \dots$, are calculated in a recurrent way

$$\phi(t) = \begin{cases} I & t = 0 \\ (A_f + \alpha I)\phi(t-1) - \sum_{j=2}^t (-1)^j \binom{\alpha}{j} \phi(t-j) & t > 0 \end{cases}$$

Proof. A state equation of the system in Eqn. (1) can be presented in the following form (Monje et al., 2010)

$$\begin{aligned} x(t+1) &= (A_f + \alpha I)x(t) - \\ &\quad - \sum_{i=2}^{t+1} (-1)^i \binom{\alpha}{i} x(t+1-i) + Bu(t) \quad (14) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

The definition of controllability Gramian is connected with the minimal energy required for the transfer of the system from state $x(0) = 0$ to $x(t) = x_p$. For the discrete-time fractional-order systems with the initial condition $x(0) = 0$ and input signal in a form of the Kronecker delta $\delta(t)$, the state response is

$$\xi(t) = \begin{cases} 0 & t = 0 \\ B & t = 1 \\ (A_f + \alpha I)\xi(t-1) - \sum_{j=2}^t (-1)^j \binom{\alpha}{j} \xi(t-j) & t > 1 \end{cases}$$

The definition of observability Gramian is connected with energy produced by the state $x(0) = x_0$ in the absence of the input signal $u(t) = 0$. Based on Eqn. (14) it can be easily shown that the states of the system are as follows

$$x(t) = (A_f + \alpha I)x(t-1) - \sum_{j=2}^t (-1)^j \binom{\alpha}{j} x(t-j)$$

$$t = 1, 2, \dots$$

Taking into account the output equation of the system we arrive at

$$\eta(t) = Cx(t)$$

As before, the controllability and observability Gramians for the fractional-order systems can be calculated in the following way

$$\begin{aligned} P &= \sum_{t=0}^{\infty} \xi(t)\xi^T(t) = \sum_{t=1}^{\infty} \phi(t-1)B(\phi(t-1)B)^T \\ &= \sum_{t=0}^{\infty} \phi(t)BB^T\phi(t)^T \\ Q &= \sum_{t=0}^{\infty} \eta^T(t)\eta(t) = \sum_{t=0}^{\infty} (C\phi(t))^T(C\phi(t)) \end{aligned}$$

This immediately results in Eqn. (13), which completes the proof.

Implementation of Eqn. (13) implies the infinite number of elements $\phi(t)$, furthermore each of $\phi(t)$ requires determination of the fractional-order difference calculated from 0 to $t+1$. Therefore, just as the implementation length L for the finite-length approximation of fractional-order difference in Eqn. (3), an additional bound J for the summation process in Eqn. (13) can be introduced. This leads to approximate solutions for the controllability and observability Gramians

$$\begin{aligned} \hat{P}_{(J,L)} &= \sum_{t=1}^J \phi(t-1)BB^T\phi(t-1)^T \\ \hat{Q}_{(J,L)} &= \sum_{t=0}^J \phi(t)^T C^T C \phi(t) \end{aligned} \quad (15)$$

where

$$\phi(t) = \begin{cases} I & t = 0 \\ (A_f + \alpha I)\phi(t-1) - \sum_{j=2}^L (-1)^j \binom{\alpha}{j} \phi(t-j) & t > 0 \end{cases}$$

with $\phi(l) = 0 \forall l < 0$ and L being an implementation length.

The introduced 'fractional' controllability and observability Gramians can be used in the SVD-originated MOR procedures for reduction of fractional-order systems.

Consider a stable discrete-time commensurate fractional-order state space system as in Eqn. (1), with fractional-order difference defined in Eqn. (2). Balanced realization of that system obtained through the linear state transformation $x \rightarrow Tx$, with T calculated e.g. as in Table. 1, can be presented as follows

$$\begin{aligned} \Delta^\alpha \bar{x}(t+1) &= TA_f T^{-1} \bar{x}(t) + TBu(t) \\ y &= CT^{-1} \bar{x}(t) + Du(t) \end{aligned} \quad (16)$$

The partition of the balanced realization of the fractional-order system into two subsystems

$$\begin{aligned} \begin{bmatrix} \Delta^\alpha \bar{x}_1(t+1) \\ \Delta^\alpha \bar{x}_2(t+1) \end{bmatrix} &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u(t) \\ y &= [\bar{C}_1 \quad \bar{C}_2] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + Du(t) \end{aligned} \quad (17)$$

enables to determine the reduced fractional-order model, which for the BTA method is given as follows

$$\begin{aligned} \Delta^\alpha x_r(t+1) &= \bar{A}_{11} x_r(t) + \bar{B}_1 u(t) \\ y_r(t) &= \bar{C}_1 x_r(t) + Du(t) \end{aligned} \quad (18)$$

The SPA method considers the fractional difference $\Delta^\alpha \bar{x}_2 = 0$. Simple transformation of Eqn. (17) enables determination of the state vector \bar{x}_2 which is equal to

$$\bar{x}_2(t) = -\bar{A}_{22}^{-1} \bar{A}_{21} \bar{x}_1(t) - \bar{A}_{22}^{-1} \bar{B}_2 u(t)$$

Therefore, the reduced model is given in the following way

$$\begin{aligned} \Delta^\alpha x_r(t+1) &= A_r x_r(t) + B_r u(t) \\ y_r(t) &= C_r x_r(t) + D_r u(t) \end{aligned} \quad (19)$$

where

$$\begin{aligned} A_r &= \bar{A}_{11} - \bar{A}_{12} \bar{A}_{22}^{-1} \bar{A}_{21} \\ B_r &= \bar{B}_1 - \bar{A}_{12} \bar{A}_{22}^{-1} \bar{B}_2 \\ C_r &= \bar{C}_1 - \bar{C}_2 \bar{A}_{22}^{-1} \bar{A}_{21} \\ D_r &= \bar{D} - \bar{C}_2 \bar{A}_{22}^{-1} \bar{B}_2 \end{aligned}$$

Remark 1. Possible accounting for the sampling period h (when transferring from a continuous-time derivative to the discrete-time difference) results in the substitutions $A_f \rightarrow A_f h^\alpha$ and $B \rightarrow B h^\alpha$ in all the relevant equations presented in Sections 2 and 3.

4. SIMULATION EXAMPLES

Consider the discrete-time fractional-order state space system as in Eqn. (1), with $\alpha = 0.85$ and

$$\left[\begin{array}{c|c} A_f & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{cccc|c} 2.37 & -4.3849 & 2.602023 & -0.5886251 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \hline 1 & -1.8 & 0.9 & 0 & 0 \end{array} \right]$$

Note that the system is asymptotically stable (see Refs. Stanisławski and Latawiec (2013a,b); Stanisławski (2017)). Using Eqn. (15) we can calculate the approximations of the controllability and observability Gramians, which for $J = L = 10^4$ are equal to

$$\begin{aligned} P &\cong \hat{P}_{(10^4, 10^4)} = \begin{bmatrix} 3458.7 & 3376.5 & 3260.0 & 3122.8 \\ 3376.5 & 3342.1 & 3267.7 & 3163.3 \\ 3260.0 & 3267.7 & 3235.2 & 3167.6 \\ 3122.8 & 3163.3 & 3167.6 & 3136.6 \end{bmatrix} \\ Q &\cong \hat{Q}_{(10^4, 10^4)} = \begin{bmatrix} 32.9587 & -78.1521 & 65.7265 & -18.6911 \\ -78.1521 & 185.7217 & -156.6265 & 44.7161 \\ 65.7265 & -156.6265 & 132.5800 & -38.057 \\ -18.6911 & 44.7161 & -38.057 & 11.0138 \end{bmatrix} \end{aligned}$$

The corresponding Hankel singular values of the system are $\{19.5765, 1.1444, 0.8791, 0.8553\}$. It can be seen that the last three values are significantly, but not negligibly, lower than the first one. On this basis, the orders of the reduced models are chosen as $k = 1$ and $k = 2$. The frequency responses both for the original system and reduced models, as well as the approximation errors are presented in Fig. 1. The obtained results confirm that the BTA method does not properly approximate the steady-state and low-frequency properties of the system. Much better approximations for low frequencies are obtained by using the SPA method, however the side effect is that we may obtain the nonzero matrix D_r , even in case the original D is zero. The issue will be a subject of our future research. It is worth mentioning that the stability preservation property for both BTA and SPA methods is confirmed in our simulation example; formal stability preservation results will also be a subject of our future research.

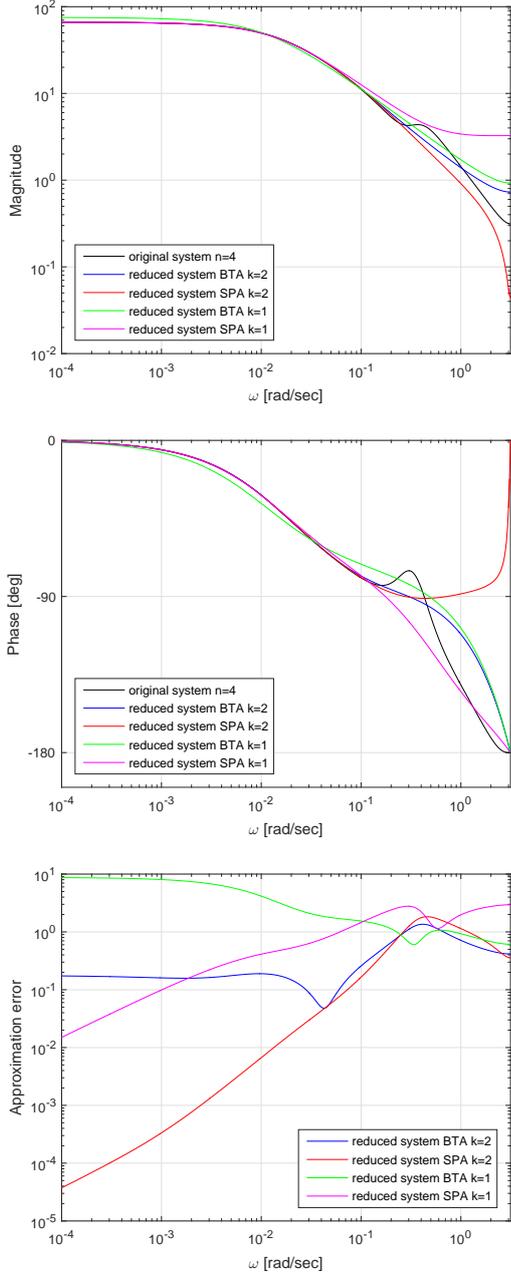


Fig. 1. Frequency responses and approximation errors for the reduced models

Practical implementation of controllability and observability Gramians, $\hat{P}_{(J,L)}$ and $\hat{Q}_{(J,L)}$, respectively, requires computation of J matrices $\phi(t)$ and approximation of fractional difference using the implementation length L for each matrix $\phi(t)$. The chosen values J and L have an impact on the the approximation accuracy for determination of controllability and observability Gramians as well as on the computational burden of the algorithm. Fig. 2 shows the error norms for calculation of controllability and observability Gramians as a function of J and L

$$\left\| P - \hat{P}_{(J,L)} \right\|_2 = \max(\text{svd}(\hat{P}_{(10^4, 10^4)} - \hat{P}_{(J,L)})) \quad (20)$$

As seen from Fig. 2, the approximation accuracy almost identically depends on the number J of the calculated matrices $\phi(t)$ and the implementation length L of FFD.

It is important to note that using $J > L$ or $J < L$ leads to the similar accuracy of Gramian approximations as in the case of $L = J = \min(L, J)$. Therefore, the best choice from the numerical point of view is assumption that $L = J$. It is important to note that accurate approximation of the FD usually requires high implementation lengths L , in particular for lower values of fractional orders α . For that reason, both parameters require to be chosen relatively high.

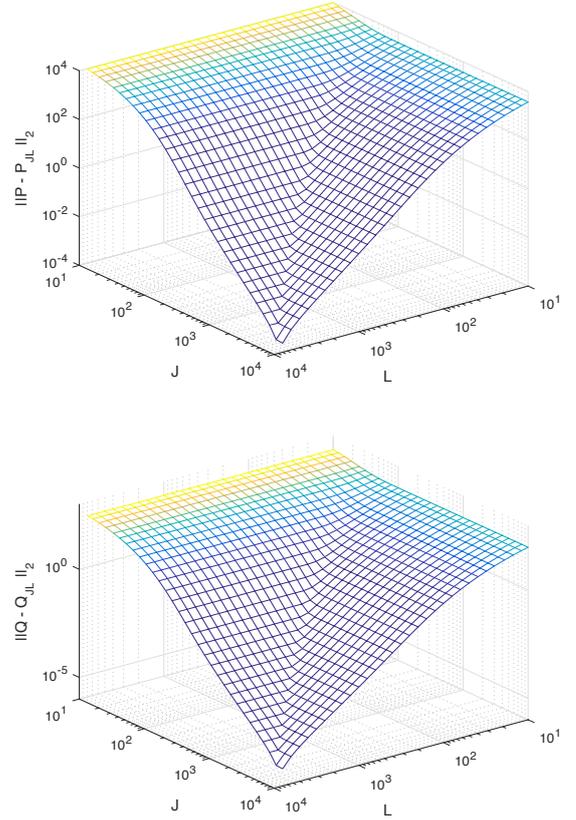


Fig. 2. Approximation accuracy for controllability and observability Gramians

5. CONCLUSIONS

This paper presents new results in SVD-based model order reduction of discrete-time commensurate fractional-order systems. The main contribution of the paper is the introduction of the controllability and observability Gramians for the fractional-order systems. The result enables implementation of the selected model order reduction methods in particular the BTA and SPA ones, to fractional-order systems. As a result we obtain low-dimension, fractional-order approximators. Simulation results confirm the effectiveness of the introduced methodology for model order reduction of discrete-time fractional-order systems.

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